A Note on Quantum Cloning in d dimensions

Paolo Zanardi *
Institute for Scientific Interchange Foundation, Villa Gualino
Viale Settimio Severo 65, I-10133 Torino, Italy
and Unità INFM, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

The quantum state space $\mathcal{S}$ over a d-dimensional Hilbert space is represented as a convex subset of a $D-1$-dimensional sphere $S_{D-1}\subset \mathbf{R}^D$, where $D=d^2-1$. Quantum tranformations (CP-maps) are then associated with the affine transformations of $\mathbf{R}^D$, and $N\mapsto M$ *cloners* induce polynomial mappings. In this geometrical setting it is shown that an optimal cloner can be chosen covariant and induces a map between reduced density matrices given by a simple contraction of the associated D-dimensional Bloch vectors.

I. INTRODUCTION

The quantum *no-cloning* theorem [1] represents the most basic difference between quantum and classical information theory. It stems simply from the unitary character of any allowable evolution for a closed quantum system. Since *perfect* copying of quantum information is forbidden it is a relevant (conceptually as well as practically) question to ask how close one can get to that ideal (unphysical) process, and in what way. More formally one has to face a complex optimization problem involving all allowed quantum transformations between multipartite Hilbert spaces (CP maps, [2]).
Several papers, addressing this issue, have appeared recently. Optimal fidelities and explicit forms for the cloning transformations have been found [3], [4], [5], [6] [7], and connections with the Quantum State Estimation problem has been made [8]. These works are mainly focused on *qubit* (i.e. bi-dimensional) systems (notably with the exception of reference [9] from which this note was inspired). In this paper a few simple results are reported about the cloning problem for an arbitrary d-dimensional quantum system, mostly obtained in a geometric framework (Generalized Bloch representation). Although no explicit computations of cloning machines or cloning fidelities [9] appear, we believe that the approach presented here deserves attention, in that it allows to rigorously generalize results obtained in $d=2$ (partly by heuristic arguments and direct calculations) and at the same time it provides a novel insight of the algebraic-geometric structure underlying the optimal quantum cloning problem.

II. CLONERS

In this section some mathematical aspects of quantum states and quantum transformations of a d-dimensional quantum system will be discussed. In particular the optimization problem of imperfect cloning will be formulated in geometric fashion.

A. The GB Representation

Let $\mathcal{H}$ be a d-dimensional Hilbert space. The set $\mathrm{End}(\mathcal{H})$ of linear operators over $\mathcal{H}$ can be endowed with a metric structure in several ways. In view of its direct connection with the geometrical framework of this paper, we shall consider $\mathrm{End}(\mathcal{H})$ as a metric space with distance

$$d(A,\,B)=2^{-1/2}\sqrt{(A-B,\,A^\dagger-B^\dagger)} \tag{1}$$

induced by the Hilbert-Schmidt scalar product $(A,\,B)\equiv \mathrm{tr}A\,B^\dagger$ (the normalization has been chosen for later convenience). The Lie algebra of hermitian $d\times d$ traceless matrices, $su(d)$, is a D-dimensional *real* subspace of $\mathrm{End}(\mathcal{H})$, where $D=d^2-1$. One can choose a basis $\{\tau_i\}_{i=1}^D$ of $su(d)$ satisfying the relations $(\tau_i,\,\tau_j)=2\,\delta_{ij}$. The set $\mathcal{B}_1$ of the unit-trace Hermitean operators is a D-dimensional hyperplane of $\mathrm{End}(\mathcal{H})$. Any element of $\mathcal{B}_1$ can be written as

$$\rho(\lambda)=\frac{1}{d}\,\mathbf{I}+\frac{1}{2}\sum_{i=1}^D\lambda_i\tau_i \tag{2}$$

The vector $\lambda\equiv(\lambda_1,\ldots,\lambda_D)\in\mathbf{R}^D$ will be referred to as the *Generalized Bloch Representation* [GBR] of ρ. Equation (2) defines a mapping $m\colon\mathcal{B}_1\to\mathbf{R}^D$ that associates to any $\rho\in\mathcal{B}_1$ its GBR vector, such that $\rho=\rho(m(\rho))$. Let $\mathcal{P}\subset\mathcal{B}_1$ the set of pure states on $\mathcal{H}$, and $\mathcal{S}=hull(\mathcal{P})$ its convex hull (the *state space*).The corresponding objects over $\mathcal{H}^{\otimes N}$ will be denoted by same notation with an extra index N.
In the following $\mathbf{R}^D$ will be considered endowed with the geometrical structure associated with the *euclidean* scalar

*zanardi@isiosf.isi.it

product $\langle x, y\rangle \equiv \sum_{i=1}^D x_i\, y_i$, and norm $\|x\| \equiv \sqrt{\langle x, x\rangle}$. Let $S_{D-1} \subset \mathbf{R}^D$ be the $(D-1)$-dimensional hypersphere with radius $R_d \equiv \sqrt{2\,(1-1/d)}$, and B_D the ball $\partial B_D = S_{D-1}$. For $d=2$ (the *qubit* case) one finds $R_2 = 1$; B_2 is the Bloch sphere. The basic properties of the GBR mapping m are collected in the following

Proposition 1 i) m is an affine bijection; ii) $(\sigma, \rho) = d^{-1} + 2^{-1}\langle m(\rho), m(\sigma)\rangle$, and $d(\rho,\sigma) = 2^{-1}\|m(\rho) - m(\sigma)\|$; iii) $m(\mathcal{P}) \subset S_{D-1}$, and $m(\mathcal{S}) \subset B_D$.

Proof

i) In order to prove affinity one has to check that $m(\mu\rho_1 + (1-\mu)\,\rho_2) = \mu\, m(\rho_1) + (1-\mu)\,\rho_2, \forall \rho_1, \rho_2 \in \mathcal{S},\, 0 \le \mu \le 1$. Since the components of m are given by $m_i(\rho) = (\tau_i, \rho)$, $(i = 1, \ldots, D)$ this is immediate. Bijectivity follows from the next point. ii) Derives by straightforward calculation using orthogonality of the τ_i's.. iii) If $\rho \in \mathcal{P}$ one has $\rho^2 = \rho$, then (by previous point) $1 = \mathrm{tr}\rho^2 = 1/d + 1/2\,\|\lambda\|^2$, whence $m(\rho) \in S_{D-1}$. For general states of $\mathcal{S}$ one has – due to affinity of m – $m(\mathcal{S}) = m(hull(\mathcal{P})) = hull(m(\mathcal{P})) \subset hull(S_{D-1}) = B_D$. $\Box$

It is important to notice that, for $d > 2$, $m(\mathcal{S})$ *is a proper subset of* B_D.

Indeed: if $\rho \in \mathcal{S} \Rightarrow \mathrm{tr}(\sigma\,\rho) \ge 0, \forall \sigma \in \mathcal{P}$, but $\mathrm{tr}(\sigma\,\rho) = 1/d + 1/2\,\langle m(\sigma), m(\rho)\rangle$, then $\|m(\sigma)\|\,\|m(\rho)\|\cos\theta \equiv \langle m(\sigma), m(\rho)\rangle \ge -2\,d^{-1}$, and since $\|m(\rho)\| \le \|m(\sigma)\| = 2\,(1-1/d)$ one has $\cos\theta \ge (1-d)^{-1}$. This constraint is automatically fulfilled for all the elements of B_D just for $d=2$. In the general case one has a maximum allowed 'angle' $\theta_M(d) = \cos^{-1}[1/(1-d)]$ Notice that $\theta_M(\infty) = \pi/2$. For example suppose $\rho(\lambda) \in \mathcal{P}$: then $(\rho(\lambda), \rho(-\lambda)) = d^{-1} - 2^{-1}\,R_d^2$, a quantity which is nonnegative just for $d \le 2$.

We recall that: i) A mapping $T\colon \mathcal{B}_1 \mapsto \mathcal{B}_1$ is referred to as *positive* if $T(\mathcal{S}) \subset \mathcal{S}$ i.e., it preserves positivity ; ii) An *affine* mapping $T\colon \mathcal{B}_1 \mapsto \mathcal{B}_1$ is referred to as *completely positive* if $\forall n$ the (trivially extended) maps

$$T_n\colon \mathcal{B}_1 \otimes \mathrm{End}(\mathbf{C}^n) \mapsto \mathcal{B}_1 \otimes \mathrm{End}(\mathbf{C}^n) \tag{3}$$

given by $T_n = T \otimes \mathbf{Id}$ are positive [2]. The set of positive [completely positive] maps of a subset $X \subset \mathcal{B}_1$ into itself, will be denoted by $\mathcal{M}(X)$ $[CP(X)]$.

The GBR can be naturally lifted to the space $\mathcal{M}(\mathcal{B}_1)$ of (not necessarily affine) *positive* mappings of $\mathcal{B}_1$ into itself by the formula $T \to \mathcal{T} = m \circ T \circ m^{-1}$, or, equivalently, by the following commutative diagram

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{T} & \mathcal{B}_1 \\ \downarrow m & & \downarrow m \\ \mathbf{R}^D & \xrightarrow{\mathcal{T}} & \mathbf{R}^D \end{array}$$

The next proposition shows that CP-maps will be associated with *affine transformations* in $\mathbf{R}^D$.

Proposition 2 Let $T \in \mathcal{M}(\mathcal{B}_1)$ be a trace-preserving CP-map. Then: i) $T(\mathbf{I}/d) = \mathbf{I}/d + \sum_j c_j \tau_j$; ii) $T(\tau_i) = \sum_{j=1}^D M_{ji}\tau_j$.

Proof

i) $T(\mathbf{I}/d)$ must be a trace one hermitian operator by definition of CP-map. ii) $T(\tau_i)$ must be traceless and hermitian; the statement follows from the fact that $\{\tau_i\}$ are a $su(d)$ basis.

Therefore, if ρ has the form (2),

$$\begin{aligned} T(\rho) &= \frac{1}{d}\,\mathbf{I} + \frac{1}{2}\sum_{i,j=1}^D \lambda_i M_{ji}\tau_j \\ &= \frac{1}{d}\,\mathbf{I} + \frac{1}{2}\sum_{j=1}^D \lambda'_j\,\tau_j, \end{aligned} \tag{4}$$

where $\lambda' = \mathbf{M}(\lambda) + \mathbf{c}$, $\mathbf{M} = (M_{ij}) \in \mathrm{End}(\mathbf{R}^D)$, $\mathbf{c} \in \mathbf{R}^D$. $\Box$

This realizes an (affine) mapping M *between the trace-preserving CP-maps on* $\mathcal{B}_1$ *and the affine transformations of* $\mathbf{R}^D$ *in itself.* $M\colon CP(\mathcal{B}_1) \to \mathit{Aff}(\mathbf{R}^D)\colon T \to M(T) = m \circ T \circ m^{-1}$.

A particularly relevant class of CP-maps is given by the unitary transformations. Any $X \in SU(d)$ defines, via the adjoint action, a CP-map on $\mathcal{B}_1$, $\rho \to \mathrm{Ad}X(\rho) \equiv X\,\rho\,X^\dagger$. The following proposition shows that unitary transformations correspond, in the GBR, to rotations.

Proposition 3 $\varphi \equiv M \circ Ad$ *is a homomorhism of* $SU(d)$ *in* $SO(D)$.

Proof

First observe that from $\mathrm{Ad}X(\mathbf{I}) = \mathbf{I}$, $(\forall X \in SU(d))$ there follows that $\mathbf{c} = 0$: $M(T)$ *is linear.* Since obviously $M(T_1\,T_2) = M(T_1)\,M(T_2)$, one has just to check that, for any $X \in SU(d)$, the mapping $\lambda \to m(\mathrm{Ad}X(m^{-1}(\lambda)))$ preserves scalar product (and then the norm) on $\mathbf{R}^D$. Indeed $\langle\lambda, \mu\rangle = 2\,(\mathrm{tr}[\rho(\lambda)\,\rho(\mu)] - 1/d)$, and the trace is Ad-invariant. Since $X\,\tau_i\,X^\dagger = \sum_j X_{ji}\tau_j$, one has that the induced $\mathbf{R}^D$ mapping has the form $\lambda \to \mathbf{X}(\lambda)$ where the matrices $\mathbf{X} = (X_{ji})$ are the adjoint representatives of $SU(d)$. $\Box$

Since $SU(d)$ acts (via Ad) *transitively* [10] on $\mathcal{P}$, it follows immediately that the subgroup $\varphi(SU(d))$ acts transitively over $m(\mathcal{P})$.

Once again it is worth emphasizing that, for $d > 2$, $\varphi(SU(d))$ *is a proper subset of* $SO(D)$. This can be easily understood observing that any pair λ, μ of points of S_{D-1} are connected by an orthogonal transformation $R_{\lambda,\mu}$; in particular one can have $\lambda \in m(\mathcal{P})$ and $\mu \notin m(\mathcal{P})$. Since $m(\mathcal{P})$ is $SU(d)$-invariant, $R_{\lambda,\mu} \notin \varphi(SU(d))$.

B. Optimality

The metric structure over $\mathcal{S}$ allows us to introduce several natural 'figures of merit' for cloning. For example let us consider, for given $T \in \mathcal{M}(\mathcal{B}_1)$, the functional $F_1\colon \mathcal{S} \to [0, 1]$ given by

$$F_1(T,\,\rho) = 1 - [d(\rho,\,T(\rho))]^2$$
$$= 2^{-1}\,\delta(T(\rho)) + F(T,\,\rho), \quad (5)$$

where $\delta(T(\rho)) \equiv 1 - \mathrm{tr}\,T^2(\rho)$ is the *idempotency deficit* (or linear entropy) of $T(\rho)$ and $F(T,\,\rho) \equiv (T(\rho),\,\rho)$, is the *pure state fidelity* [11].

The naturality of F_1 as (state dependent) measure of cloning goodness should be clear: it is maximum (equal to 1) when $\rho = T(\rho)$ and minimum (0) when ρ and $T(\rho)$ have disjoint supports. Moreover both contributions δ and F to the 'merit' function F_1 have a clear geometrical meaning in $\mathbf{R}^D$. Indeed, by using the GBR one finds (from Proposition 1)

$$\delta(T(\rho)) = \frac{1}{2}\,(R_d^2 - \|\mathcal{T}(\lambda)\|^2),$$
$$F(T,\,\rho) = \frac{1}{d} + \frac{1}{2}\,\langle \mathcal{T}(\lambda),\,\lambda\rangle. \quad (6)$$

It is interesting to consider a special class of transformations for which the quality of the cloning process is independent on the (pure) input state [7]. This motivates the following

Definition 1 A map $T \in \mathcal{M}(\mathcal{S})$ is *universal* if $F_1(T,\rho)$ is independent on $\rho \in \mathcal{P}$.

For general maps (i.e. non universal) one can be interested in optimizing the worst case, with *pure* initial input. Therefore it is natural to introduce the quantity

$$\tilde{F}_1(T) = \min_{\rho\in\mathcal{P}}\; F_1(T,\,\rho). \quad (7)$$

The following proposition will turn to be useful:

Proposition 4 i) *$\tilde{F}_1$ is a concave functional over $\mathcal{M}(\mathcal{S})$.* ii) If $U \in SU(d)$ and $T_U \in \mathcal{M}(\mathcal{S})$ is defined by $T_U(\rho) = U^\dagger\,T(U\,\rho\,U^\dagger)\,U$, one has $\tilde{F}_1(T_U) = \tilde{F}_1(T)$.

Proof

i) Let $T_1,\,T_2 \in \mathcal{M}(\mathcal{S}), \mu \in \mathbf{R}_0^+$. In view of the concavity of δ one has $F_1(\mu\,T_1 + (1-\mu)\,T_2,\,\rho) \ge 2^{-1}\,\mu\,\delta(T_1(\rho)) + 2^{-1}\,(1-\mu)\,\delta(T_2(\rho)) + \mu\,F_1(T_1,\,\rho) + (1-\mu)\,F_1(T_2,\,\rho)$. Then, by the superadditivity of the infimum one gets

$$\tilde{F}_1(\mu\,T_1 + (1-\mu)\,T_2) \ge \mu\,\tilde{F}_1(T_1) + (1-\mu)\,\tilde{F}_1(T_2).$$

ii) Explicitly using $SU(d)$-invariance of the metric, and transitivity of $SU(d)$-action over $\mathcal{P}$,

$$\begin{aligned}\tilde{F}_1(T_U) = &\inf_{\rho\in\mathcal{S}} F(T_U,\,\rho) = \\ &\inf_{\rho\in\mathcal{S}} (1 - d^2(\rho,\,U^\dagger\,T(U\,\rho\,U^\dagger)\,U)) \\ = &\inf_{\sigma\in\mathcal{S}} (1 - d^2(U^\dagger\,\sigma\,U,\,U^\dagger\,T(\sigma)\,U)) \\ = &\inf_{\sigma\in\mathcal{S}} (1 - d^2(\sigma,\,T(\sigma)) = \tilde{F}_1(T).\end{aligned}$$

$\square$

The mapping $T \to T_U$ defines a $SU(d)$-action Φ such that $T_U \equiv \Phi(U,\,T)$ over $\mathcal{M}(\mathcal{S})$. Point ii) of the previous proposition simply states that the quality of cloning is constant along the orbits of Φ. The fixed points of Φ therefore play a special role.

Definition 2 A map $T \in \mathcal{M}(\mathcal{S})$ is *covariant* iff $T_X = T,\,\forall X \in SU(d)$.

Next proposition shows that covariance implies universality and imposes strong geometrical constraints to the GBR.

Proposition 5 *Suppose $T \in \mathcal{M}(\mathcal{S})$ covariant. Then: i) T is universal, ii) $\mathbf{U}\,\mathcal{T}_k(\lambda) = \mathcal{T}_k(\mathbf{U}\,\lambda),\ \forall\lambda \in \mathbf{R}^D,\ \mathbf{U} \in \varphi(SU(d))$; iii) $\|\mathcal{T}(\lambda)\|$ and $\langle\mathcal{T}(\lambda),\,\lambda\rangle$ are constant over $m(\mathcal{P})$*

Proof

i) Since $\mathrm{Ad}SU(d)$ is transitive over $\mathcal{P}$, it suffices to show that $F_1(T,\rho) = F_1(T,\mathrm{Ad}X(\rho))$, $(\forall X \in SU(d),\ \rho \in \mathcal{P})$. Indeed, $\delta(T(\mathrm{Ad}X\rho)) = \delta(\mathrm{Ad}X\,T(\rho)) = \delta(T(\rho))$, and

$$\begin{aligned}&F(\rho,T) = \mathrm{tr}\,(\rho\,T(\rho)) = \mathrm{tr}\left(X^\dagger\,\rho\,T(\rho)\,X\right) = \\ &\mathrm{tr}\left(X^\dagger\,\rho\,X\,T(X^\dagger\,\rho\,X)\right) = \mathrm{tr}\,(\mathrm{Ad}X(\rho)\,T(\mathrm{Ad}X(\rho)))\,. \quad (8)\end{aligned}$$

ii) This point requires just an explicit check. iii) If $\lambda,\,\lambda' \in m(\mathcal{P}) \Rightarrow \exists\mathbf{U} \in \varphi(SU(d)),\ s.t.\ \mathbf{U}\lambda = \lambda'$. Then $\|\mathcal{T}(\lambda')\| = \|\mathcal{T}(\mathbf{U}\lambda)\| = \|\mathbf{U}\mathcal{T}(\lambda)\| = \|\mathcal{T}(\lambda)\|$. Moreover, $\forall\lambda,\lambda' \in m(\mathcal{P})$,

$$\begin{aligned}&\langle\mathcal{T}(\lambda),\,\lambda\rangle = \langle\mathbf{U}\mathcal{T}(\lambda),\,\mathbf{U}\lambda\rangle = \\ &\langle\mathcal{T}(\mathbf{U}\lambda),\,\mathbf{U}\lambda\rangle = \langle\mathcal{T}(\lambda'),\,\lambda'\rangle. \quad (9)\end{aligned}$$

$\square$

Mappings satisfying relation ii), for $\mathbf{U}$ belonging to some group $\mathcal{G}$, are known as *$\mathcal{G}$-automorphic functions*. Therefore point ii) of the previous proposition can be rephrased saying that *GBR of covariant maps of $\mathcal{M}(\mathcal{S})$ are $\mathcal{G}$-automorphic functions of $\mathbf{R}^D$ in itself, where $\mathcal{G} \equiv \varphi(SU(d))$.*

Of course any linear mapping $M \in \mathrm{End}(\mathbf{R}^D)$ is $\mathcal{G}$-automorphic for *any* subgroup $\mathcal{G} \subset GL(D,\,\mathbf{R})$ such that $[M,\,\mathcal{G}] = 0$ (M belongs to the *centralizer* of $\mathcal{G}$). An example of $SO(D)$-automorphic functions is given by $\mathcal{T}(\lambda) = f(\|\lambda\|)\,\lambda$, with $f\colon \mathbf{R} \to (0,\,1)$. Notice that, for these mappings, the functions (5) are constants over $D-1$-dimensional spheres.

Let $\mathcal{M}'$ a convex Φ-invariantt subset of $\mathcal{M}(\mathcal{S})$ The notion of optimality used in this paper is given by

Definition 3 Let $\mathcal{M}'$ a convex Φ-invariantt subset of $\mathcal{M}(\mathcal{S})$. A map $T^* \in \mathcal{M}'$ is *optimal* (in $\mathcal{M}'$) if

$$\tilde{F}_1(T^*) = \sup_{T\in\mathcal{M}'} \tilde{F}_1(T). \quad (10)$$

Now we show that, *as far as optimality is concerned, one can restrict oneself to covariant transformations without loss of generality.* The basic idea is very simple: since our 'merit' functional F_1 is concave and $SU(d)$-invariant one can be build, for any given $SU(d)$-orbit, a convex average transformation T^* non-decreasing the cloning quality (i.e. $\tilde{F}_1(T^*) \ge \tilde{F}_1(T)$). T^* will be, by construction, covariant and it is clear that the one associated with an optimal cloner will turn out to be optimal as well.

Proposition 6 *The optimal map can be chosen to be covariant.*

Proof

Given the $SU(d)$ action Φ over $\mathcal{M}'$, the element of $\mathcal{M}'$, $T^* = \int_{SU(d)} d\mu(X) T_X$ (where $T_X = \Phi(X, T)$, for a given $T \in \mathcal{M}'$) is a covariant map. Indeed, for any $Y \in SU(d)$, $T^*_Y = \int_{SU(d)} d\mu(X)(T_X)_Y = \int_{SU(d)} d\mu(X) T_{XY} = \int_{SU(d)} d\mu(XY) T_{XY} = T^*$, where the invariance of the Haar measure $d\mu$ was used [12]. If T is optimal $\tilde{F}_1(T) \geq \tilde{F}_1(T^*) \geq \int_{SU(d)} d\mu(X) \tilde{F}_1(T_X) = \tilde{F}_1(T)$; thus $\tilde{F}_1(T^*) = \tilde{F}_1(T)$ (where we used the concavity of $\tilde{F}_1$, Proposition 4, and the normalization $\int_{SU(d)} d\mu(X) = 1$ of the measure $d\mu$). □

Notice that if T is universal, then the mapping T^* introduced in Proposition 6 has the same value of the merit functional. Indeed $\forall \rho \in \mathcal{P}$ one has

$$\begin{aligned} \tilde{F}_1(T) = F_1(T,\, \rho) = \int d\mu(X) F_1(T,\, X\,\rho\, X^\dagger) \\ = \int d\mu(X)(1 - d^2(X\,\rho\, X^\dagger,\, T(X\,\rho\, X^\dagger))) \\ = \int d\mu(X)(1 - d^2(\rho, X^\dagger\, T(X\,\rho\, X^\dagger)\, X)) \\ = 1 - d^2(\rho,\, T^*(\rho)) = \tilde{F}_1(T^*), \end{aligned} \tag{11}$$

where we have used linearity in T of F_1, normalization of $d\mu$ and $SU(d)$-invariance of the metric.

This observation makes clear that for optimization purposes t one can identify the notion of covariance and universality: *a covariant map is universal and for any universal map there exists a covariant map with same cloning quality.*

The next theorem shows that the structure of *affine* covariant maps is very simple.

Proposition 7 *$T \in CP(\mathcal{S})$ is covariant iff $\mathcal{T} = \xi\, \mathbf{I}$ with $\xi \in (0,\, 1)$.*

Proof

a) If $\mathcal{T} = \xi\, \mathbf{I}$, it is trivial to check that T is covariant. b) The components of the GBR of T are given by $\mathcal{T}_i(m(\rho)) = (\tau_i,\, T(\rho)) = (F_i,\, \rho)$, where $F_i = T^t(\tau_i)$ are traceless operators (T^t is the transpose map of T with respect to the Hilbert-Schmidt scalar product). On has to show that $F_i = \xi\, \tau_i$. If T is covariant, $T(X\,\rho\, X^\dagger) = X\, T(\rho)\, X^\dagger$; therefore

$$\begin{aligned} (\tau_i,\, T(X\,\rho\, X^\dagger)) = (F_i,\, X\,\rho\, X^\dagger) = (X^\dagger\, F_i\, X,\, \rho) = \\ (\tau_i,\, X\, T(\rho)\, X^\dagger) = (X^\dagger\, \tau_i\, X,\, T(\rho)) = \\ \sum_j X_{ji}\, (\tau_j,\, T(\rho)) = \sum_j X_{ji}\, (F_j,\, \rho). \end{aligned} \tag{12}$$

Since this equality holds for any $\rho \in \mathcal{S}$, one gets

$$X^\dagger\, F_i\, X = \sum_j X_{ji}\, F_j, \tag{13}$$

namely the F_i's transform under the adjoint action of $SU(d)$ as the τ_i's. As such action is *irreducible*, this implies that $F_i = \xi\, \tau_i$, $(i = 1, \ldots, D)$. Indeed let $F_i = \sum_j M_{ij}\, \tau_j$,: from equation (13) one finds $[\mathbf{X},\, \mathbf{M}] = 0$, $\forall X \in SU(d)$, then – by Schur's lemma – $\mathbf{M} = \xi\, \mathbf{I}$. Moreover $\xi \in [0,\, 1]$ due to positivity requirement. □

A covariant map $T \in CP(\mathcal{S})$ has the form [8] given by

$$T(\rho) = (1 - \xi)\, d^{-1}\, \mathbf{I} + \xi\, \rho. \tag{14}$$

The following example shows that one can have a whole family of covariant, positive, trace-preserving, *non-linear* maps of $\mathcal{S}$ in itself $T_\Gamma(\rho) = (1 - \Gamma[\rho])\, d^{-1}\, \mathbf{I} + \Gamma[\rho]\, \rho$, where $\Gamma\colon \mathcal{S} \to (0,\, 1)$ is a $SU(d)$-invariant non-linear functional. Such functionals can be built, for example, given any non-linear map $\gamma\colon (0,\, 1) \to (0,\, 1)$ by any convex superposition of the maps $\Gamma_n(\rho) = \gamma(\mathrm{tr}\, \rho^n)$.

These maps, restricted to $\mathcal{P}$, amount to a simple (state independent) shrinking of the generalized Bloch vector $m(\rho)$. Nevertheless, since they are not affine, the property cannot be extended to the whole $\mathcal{S}$.

III. CLONERS $N \to M$

Now we turn the $N \mapsto M$ cloning. In this section we shall set $\tau_0 \equiv \mathbf{I}$, $\lambda_0 \equiv d^{-1}$, and $\lambda_i \mapsto \lambda_i/2$. Let us consider the N-system state $\rho_N \equiv \rho(\lambda)^{\otimes N}$,

$$\rho_N = \sum_{i_1, \ldots, i_N = 0}^{D} \lambda_{i_1} \cdots \lambda_{i_N}\, \otimes_{k=1}^{N} \tau_{i_k} = \sum_{\mathbf{i} \in \mathcal{F}_{N,D}} \lambda_{\mathbf{i}}\, \tau_{\mathbf{i}}, \tag{15}$$

where $\mathcal{F}_{N,D}$ is the set of the maps from $\{0, \ldots, N\}$ to $\{0, \ldots, D\}$, and $\lambda_{\mathbf{i}} \equiv \prod_{k=0}^{N} \lambda_{i_k}$, $\tau_{\mathbf{i}} \equiv \otimes_{k=0}^{N} \tau_{i_k}$. Notice that in equation (15) the only non trace-less term is $\lambda_{\mathbf{0}}\, \tau_{\mathbf{0}} \equiv d^{-N}\, \mathbf{I}^{\otimes N}$.

The set of trace-preserving CP-maps from $\mathcal{S}^N$ to $\mathcal{S}^M$ will be denoted as $CP_{M,N}$.

The problem is now to find the optimal (with respect to some some criterion) transformations of $CP_{M,N}$.

Since $X \in SU(d)$ acts naturally over $CP_{M,N}$ by $\Phi^N\colon (X,\, T) \mapsto T_X$ in which

$$T_X(\rho) = \mathrm{Ad}^{\otimes M} X^\dagger(\, T(\mathrm{Ad}^{\otimes N} X(\rho))), \tag{16}$$

the notion of covariance is immediately extended to $CP_{M,N}$. It means that T 'intertwines' between the N and M-fold tensor representations of $SU(d)$. This can be pictorially described by the following commutative diagram

$$\begin{array}{ccc} \mathcal{S}^N & \xrightarrow{\mathrm{Ad}^{\otimes N} X} & \mathcal{S}^N \\ \downarrow T & & \downarrow T \\ \mathcal{S}^M & \xrightarrow{\mathrm{Ad}^{\otimes M} X^\dagger} & \mathcal{S}^M \end{array}$$

To grasp what covariance means consider a set of operators $\{\phi_i\}_i$ in the domain of $T \in CP_{M,N}$, that under the $\mathrm{Ad}^{\otimes N}$-action of $SU(d)$ transform according to an irreducible representations $\mathbf{R}$ (i.e. $\mathrm{Ad}^{\otimes N} X(\phi_i) = \sum_j \mathbf{R}_{ji}(X)\phi_j$). If T is covariant then $\tilde{\phi}_i \equiv T(\phi_i)$ transform under $\mathrm{Ad}^{\otimes M}$ according the *same* irrep. *In other words a covariant mapping conserves the $SU(d)$ symmetry content of the states.* For example $[X^{\otimes N}, \rho] = 0 \Rightarrow [X^{\otimes M}, T(\rho)] = 0$, in particular if $\rho = d^{-1}\,\mathbf{I}$ one has that $T(\rho)$ belongs to the centralizer $\mathcal{A}_{d,M}$ of the n-fold tensor representation of $SU(d)$. $\mathcal{A}_{d,M}$ is an algebra generated by the representatives of the symmetric group S_N acting in the natural way. Of course for $\mathcal{A}_{d,1} \propto \mathbf{I}$.

In the multi-system case now under consideration, one has also the natural action of the symmetric group S_M over $\mathcal{S}^M$ [if $\sigma \in S_M$ and $\rho = |\Psi\rangle\langle\Psi|$, $\sigma\cdot\rho \equiv U_\sigma|\Psi\rangle\langle\Psi|U_\sigma^\dagger$, where $U_\sigma \otimes_{j=1}^M |\psi_j\rangle = \otimes_{j=1}^M |\psi_{\sigma(j)}\rangle$.] therefore one can consider the maps $T_\sigma(\rho) = \sigma \cdot T(\rho)\,(\sigma \in S_M)$.

Definition 4 A map $T \in CP_{M,N}$ such that $T \equiv T_\sigma,\, \forall \sigma \in S_M$ will be referred to as *symmetric.*

Remark For symmetric maps $T(\rho)$ is totally symmetric operator. Let us denote with $\mathrm{tr}_{\bar{k}}$ the trace over all but the k-th factor of the tensor product $\mathcal{H}^{\otimes M}$. One can associate, to any element $T \in CP_{M,N}$, M *reduced maps* of $\mathcal{M}(\mathcal{S})$ defined by the rule $T_k\colon \rho \to \mathrm{tr}_{\bar{k}}\, T(\rho^{\otimes N})$.

Proposition 8 *The maps $\{T_k\}_{k=1}^M$ fullfill the following i) $T_k \in \mathcal{M}(\mathcal{S})$. ii) The GBR of the T_k's have components that are polymomials of order N. iii) If T is symmetric the T_k's are identical. iv) If T is covariant so are the T_k's.*

Proof

i) The T_k's are positive in that they are compositions of the positive maps. ii) One has $T(\tau_{\mathbf{i}}) = \sum_{\mathbf{j}\in\mathcal{F}_{M,D}} M_{\mathbf{j},\mathbf{i}}\,\tau_{\mathbf{j}}$, therefore $T(\rho_N) = \sum_{\mathbf{j}\in\mathcal{F}_{M,D}} \lambda'_{\mathbf{j}}\tau_{\mathbf{j}}$, where $\lambda'_{\mathbf{j}} = \sum_{\mathbf{i}\in\mathcal{F}_{N,D}} M_{\mathbf{j},\mathbf{i}}\,\lambda_{\mathbf{i}}$. In particular $T(\tau_{\mathbf{0}}) = d^{-M}\,\mathbf{I}^{\otimes M} + \sum_{\mathbf{j}\neq\mathbf{0}} c_{\mathbf{j}}\tau_{\mathbf{j}}$, and $\mathbf{i} \neq \mathbf{0} \Rightarrow \mathrm{tr}T(\tau_{\mathbf{i}}) = 0 \Rightarrow M_{\mathbf{0},\mathbf{i}} = 0$. Moreover $\mathrm{tr}_{\bar{k}}\,\tau_{\mathbf{j}} = \tau_{j_k}\, d^{M-1}\prod_{l\neq k}\delta_{j_l,0}$. Therefore $T_k(\rho_N) = d^{-1}\,\mathbf{I} + \sum_{j=1}^D \mathcal{T}_k^j(\lambda)\,\tau_j$, where

$$
\mathcal{T}_k^j(\lambda) = d^{M-1}\sum_{\mathbf{i}\neq\mathbf{0}}(M_{\mathbf{j}_k,\mathbf{i}}\lambda_{\mathbf{i}} + c_{\mathbf{j}_k}). \tag{17}
$$

Here $\mathbf{j}_k$ is a M-component vector with j in the k-th entry and zero elsewhere. iii) If T is symmetric, it is simple to check that $M_{\mathbf{j}\circ\sigma,\mathbf{i}} = M_{\mathbf{j},\mathbf{i}}$, and $c_{\mathbf{j}\circ\sigma} = c_{\mathbf{j}},\ \forall\sigma\in S_M,\ \mathbf{i}\in\mathcal{F}_{N,D},\ \mathbf{j}\in\mathcal{F}_{M,D}$. In particular, if $l, k \in \{1,\ldots,M\}$, by applying the transposition $\sigma_{kl} = (k,l)$ one finds $\mathcal{T}(\lambda)_k^j = \mathcal{T}(\lambda)_l^j$. iv) One proves, by direct calculation, that

$$
\begin{aligned}
&T_k(X\,\rho\,X^\dagger) = \mathrm{tr}_{\bar{k}}T((X\,\rho\,X^\dagger)^{\otimes N})\\
&= \mathrm{tr}_{\bar{k}}T(X^{\otimes M}\rho^{\otimes N}\,X^{\dagger\otimes M}) = \mathrm{tr}_{\bar{k}}X^{\otimes M}T(\rho^{\otimes N})X^{\dagger\otimes M}\\
&X\,\mathrm{tr}_{\bar{k}}T(\rho^{\otimes N})X^\dagger = X\,T_k(\rho)\,X^\dagger.
\end{aligned} \tag{18}
$$

□

Definition 5 We introduce, for the elements of $CP_{M,N}$, the (global) figures of merit based on the quality of the reduced clones

$$
\begin{aligned}
F_1^{MN}(T,\,\rho) &= \min_k F_1(T_k,\,\rho)\ (\rho\in\mathcal{P}),\\
\tilde{F}_1^{MN}(T) &\equiv \min_{\rho\in\mathcal{P}} F_1(T,\,\rho),
\end{aligned} \tag{19}
$$

the notion of optimality being given as for the reduced maps for a convex, Φ^N-invariant $\mathcal{M}'_{M,N} \subset CP_{M,N}$.

The next proposition is an extension of proposition 6 to the $N \mapsto M$ case.

Proposition 9 *An optimal $T \in \mathcal{M}'_{M,N}$ can be chosen covariant and symmetric.*

Proof

Let us first observe that the functional $\tilde{F}_1^{MN}$ is constant over the orbits of both the $SU(d)$ and S_M actions. Indeed for $k = 1,\ldots,M$, $U \in SU(d)$, $\sigma \in S_M$, one has : i) $(T_k)_U = (T_U)_k$,from which $\tilde{F}_1^{MN}(T_U) = \tilde{F}_1^{MN}(T)$ and ii) $(T_\sigma)_k = T_{\sigma^{-1}(k)}$, from which $\tilde{F}_1^{MN}(T_\sigma) = \tilde{F}_1^{MN}(T)$. Furthermore, it follows from linearity of the mapping $T \mapsto T_k$, the properties of $\tilde{F}_1$, and $\min_k$ that $\tilde{F}_1^{MN}$ is a concave functional over $CP_{M,N}$. Now one can proceed as in Proposition 1, by introducing the 'covariantized' maps $T_{\mathcal{G}}^* \equiv \int_{\mathcal{G}} d\mu(g)\,T_g\,(\mathcal{G} = SU(d),\,S_M)$. [For the symmetric group the covariant map associated to T is $T^* = (M!)^{-1}\sum_{\sigma\in S_M} T_\sigma$]. □

A. Universal Cloners

Let us suppose that the map $T^{MN} \in CP_{MN}$ is defined over the input set

$$
\mathcal{S}_{in} \equiv \{\rho^{\otimes N},\,\rho\in\mathcal{P}\}. \tag{20}
$$

According to Proposition 9, such a map can be assumed – for optimality purposes – *covariant and symmetric.* The associated (reduced) pure-state fidelity, that has to be minimized over $m(\mathcal{P})$, is given by equation (5) [for a symmetric cloner $T^{MN} \in CP_{MN}$ we put $T_k^{MN} = T\,(k = 1,\ldots,M)$, whereby $\mathcal{T}\colon m(\mathcal{P}) \to m(\mathcal{S})$].

The next theorem shows how the deep geometrical meaning of covariance allows us to easily characterize the solutions of the optimization problem.

Theorem 1 *An optimal cloner $\rho \to T^{MN}(\rho)\,(\rho \in \mathcal{S}^N \bigcap \mathrm{span}\,\mathcal{S}_{in})$, can be chosen in such a way that the associated reduced map is given by a shrinking of the generalized Bloch vector.*

Proof

Due to the compacteness of $m(\mathcal{P})$, there exists a $\lambda^* \in m(\mathcal{P})$ such that $\tilde{F}(T) = 1/4\,(R_d^2 - \|\mathcal{T}(\lambda^*)\|^2) + d^{-1} + 1/2\,\langle\mathcal{T}(\lambda^*),\,\lambda^*\rangle$. Then

$$
\begin{aligned}
\tilde{F}(T) \le\ & 1/4\,(R_d^2 - \|\mathcal{T}(\lambda^*)\|^2)\\
&+ d^{-1} + 1/2\,\|\lambda^*\|\,\|\mathcal{T}(\lambda^*)\|.
\end{aligned} \tag{21}
$$

First notice that, since T can be chosen to be covariant, one has, from Proposition 5, that $\|\mathcal{T}(\lambda)\|$ is a constant

over $m(\mathcal{P})$. Therefore: i) the first contribution to the fidelity does not depend on λ, ii) the upper bound can be achieved if $\mathcal{T}(\lambda^*) = \xi\,\lambda^*$. Now we observe that, as the scalar product $\langle \mathcal{T}(\lambda),\,\lambda\rangle$ is constant over $m(\mathcal{P})$ (Proposition 5), then $\mathcal{T}(\lambda) = \xi(\lambda)\,\lambda$. But the automorphic constraint implies $\xi(\lambda) = \xi(\mathbf{U}\lambda)$, $\forall \mathbf{U} \in \varphi(SU(d))$ whence – by transitivity of the $SU(d)$-action over $m(\mathcal{P})$ – it must be $\xi\,|_{m(\mathcal{P})} = const$. The optimal (reduced) map has the form (14). Since this map is *affine*, it can be extended to the whole set of states belonging to the linear span of $\mathcal{S}_{in}$. $\square$

Remark. One must have $\mu'_j = \xi\,\lambda_j$ $(j = 1,\ldots,D)$. Therefore $M_{j0\ldots0,\mathbf{i}} = 0$ unless $\exists\, l \in \{0,\ldots,N\}$ such that $i_m = 0\,(m \neq l)$ *and* $i_l = j$. In this case one finds

$$\xi = \sum_{k=1}^{N} M(T)_{\mathbf{j}_l,\,\mathbf{i}_k},\ (l = 1,\ldots,M). \tag{22}$$

Therefore

$$T(\rho_{in}) = d^{-M}\,\mathbf{I}^{\otimes M} + N\,\xi\sum_{j=1}^{D}\lambda_j\Delta_M(\tau_i) + R(\lambda), \tag{23}$$

where $\Delta_M(\tau_i) \equiv M^{-1}\sum_{l=1}^{M}\tau_i^{(l)}$ is the *coproduct* of τ_i [*i.e.* $\tau_j^{(l)}$ acts as τ_j in the l-th factor of the tensor product and trivially in the others] and $R(\lambda)$ contains all the tensor products in which a factor $\tau_j \neq \mathbf{I}$ appears at least twice. $\square$

B. Algebraic approach

In this section we shall show that the shrinking property (14) follows from covariance alone. To this aim it is convenient to turn to a more algebraic approach in that the notion of covariance is naturally related to representation-theoretic concepts. We consider now general $\rho \in \mathcal{S}$.

Proposition 10 *The components of the map $\mathcal{T}$ are given by $\mathcal{T}_i(\lambda) = (F_i,\,\rho_\lambda^{\otimes N})$, $(i = 1,\ldots,D)$ where $F_i \in End(\mathcal{H}^{\otimes N})$ are S_N-invariant, traceless hermitian operators.*

Proof

By using S_M-invariance of T^{MN} one checks directly that the components of the map $\mathcal{T}$ are

$$\begin{aligned}\mathcal{T}_i(\lambda) &= (\tau_i,\,T(\rho)) = (\tau_i,\,\mathrm{tr}_{\bar{1}}(T^{MN}(\rho_\lambda^{\otimes N})))\\ &= \left(\tau_i\otimes\mathbf{I}^{\otimes(M-1)},\,T^{MN}(\rho_\lambda^{\otimes N})\right)\\ &= \left(\Delta_M(\tau_i),\,T^{MN}(\rho_\lambda^{\otimes N})\right) = (F_i,\,\rho_\lambda^{\otimes N}),\end{aligned} \tag{24}$$

where $F_i \equiv {T^{MN}}^t(\Delta_M(\tau_i))$. Now we observe that, since $\rho_\lambda^{\otimes N}$ is S_N-invariant, the F_i's can be chosen to symmetric

$$\begin{aligned}(F_i,\,\rho_\lambda^{\otimes N}) &= \frac{1}{N!}\sum_{\sigma\in S_N}(F_i,\,U_\sigma\,\rho_\lambda^{\otimes N}\,U_\sigma^\dagger)\\ &= \frac{1}{N!}\sum_{\sigma\in S_N}(U_\sigma^\dagger\,F_i\,U_\sigma,\,\rho_\lambda^{\otimes N}) = (\tilde{F}_i,\,\rho_\lambda^{\otimes N}),\end{aligned} \tag{25}$$

where $\tilde{F}_i \equiv 1/M!\sum_{\sigma\in S_N}U_\sigma^\dagger\,F_i\,U_\sigma$ is manifestly symmetric. Tracelessness and hermiticity follow form the general properties of CP-maps. $\square$

From the covariance constraint it follows that

$$(U^{\otimes N}\,F_i\,U^{\dagger\otimes N},\,\rho_\lambda^{\otimes N}) = \sum_{j=1}^{D}\mathbf{X}_{ji}(U)\,(F_j,\,\rho_\lambda^{\otimes N}). \tag{26}$$

By introducing the functionals Λ_ρ over $\mathrm{End}(\mathcal{H}^{\otimes N})$, $\Lambda_\rho\colon A \mapsto (\rho,\,A)$, equation (26) can be rewritten as $\Lambda_{\rho^N}(A_i^U) = 0$ $\quad(\forall\rho\in\mathcal{S},\,U\in SU(d),\,i = 1,\ldots,D)$, and

$$A_i^U \equiv U^{\otimes N}\,F_i\,U^{\dagger\otimes N} - \sum_{j=1}^{D}\mathbf{X}_{ji}(U)\,F_j. \tag{27}$$

Notice that, for $N = 1$, from (functional) equation $\Lambda_{\rho^N}(A_i^U) = 0$ follows the *operatorial* equation (13).

Let us consider now the pure state case $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle \in \mathcal{H}$. Let $\mathcal{H}_{sym}^N$the totally symmetric subspace of $\mathcal{H}^{\otimes N}$. One has: i) $\mathcal{H}_{sym}^N$ is the space associated to the identity representation of S_N; ii) it is also the space of a totally symmetric (irreducible) representation ϕ_s of $SU(d)$; iii) $\mathcal{H}_{sym}^N = \mathrm{span}\{|\psi\rangle^{\otimes N} : |\psi\rangle\in\mathcal{H}\}$.

Theorem 2 *A covariant cloner over $\mathcal{H}_{sym}^N$ induces a mapping between reduced states given by a simple shrinking of the generalized Bloch vector.*

Proof

It follows from i)–iii) that *the linear span of the operators $\rho^{\otimes N}$ $(\rho\in\mathcal{P})$ is the space of states with support in* $\mathcal{H}_{sym}^N$ [9]. In this case – since a symmetric operator leaves $\mathcal{H}_{sym}^N$ invariant – the A_i^U's can be considered as belonging to $\mathrm{End}(\mathcal{H}_{sym}^N)$. The functional equations $\Lambda_\rho(A_i^U) = 0$ then imply the operatorial equations $A_i^U = 0$ over $\mathcal{H}_{sym}^N$. Let Φ be the representation over $\mathrm{End}(\mathcal{H}_{sym}^N)$ associated with ϕ_s. Since $\mathrm{End}(\mathcal{H}_{sym}^N) \cong \mathcal{H}_{sym}^N\otimes\mathcal{H}_{sym}^{N*}$ one has $\Phi \cong \phi_s\otimes\phi_s^*$, the tensor product or two totally symmetric $SU(d)$-irreps, therefore in the decomposition of Φ *each $SU(d)$-irrep appears once* [12]. As $A_i^U = 0$ simply means that the F_i's transform according to the adjoint representation, one must have $F_i \equiv {T^{MN}}^t(\Delta_M(\tau_i)) = \xi\,\Delta_N(\tau_i)$. Form this relation it follows (see equation (24)) that $\mathcal{T}_i(\lambda) = \xi\,\lambda_i\,(i = 1,\ldots,D)$. $\square$

This proof helps to shed some light on the basic difference between the pure and the general (mixed) state problem. Let $\mathcal{H}^{\otimes N} = \oplus_{j\in\mathcal{J}}\mathcal{H}^{(j)}$ denote the decomposition of the input Hilbert space into S_N-isotopical components (i.e. $\mathcal{H}^{(j)}$ is the subspace of vectors transforming according a given S_N-irrep labelled by j). If Π_j denotes the projector over $\mathcal{H}^{(j)}$ one has, for general ρ that $\rho^{\otimes N} = \sum_{j\in\mathcal{J}}\lambda_j\rho_N^{(j)}$,

where $\rho_N^{(j)} \equiv \lambda_j^{-1}\,\Pi_j\rho^{\otimes N}\Pi_j,\ \lambda_j \equiv \mathrm{tr}(\rho^{\otimes N}\,\Pi_j)$. In this case the relevant functional equations from covariance are

$$\sum_{j\in\mathcal{J}} \mathrm{tr}(\rho_N^{(j)}\,A_i^U) = 0, \tag{28}$$

$i = 1,\ldots,D,\ U \in SU(d)$, where in each term A_i^U can be considered as belonging to $\mathrm{End}(\mathcal{H}^{(j)})$. When $\rho \in \mathcal{P}$ only the $j = 0$ ($\mathcal{H}^{(0)} \equiv \mathcal{H}_{sym}^N$) term survives, and one succeeds in getting an operatorial equation. In general one has to deal directly with equations (28), that represent a much weaker constraint on the cloner structure.

The next (almost obvious) corollary shows that concatenating optimal cloners the shrinking factors multiply

Corollary 1 *Let $T_1 \in CP_{M,N}$ and $T_2 \in CP_{R,M}$ be symmetric and covariant maps. Then: i) $T_2\circ T_1$ is a covariant and symmetric map. ii) Let $r(T)$ denote the unique map of $\mathcal{M}(\mathcal{S})$ associated to a symmetric $T \in CP_{M,N}$. If $r(T)$ is affine then $r(T_2\circ T_1) = r(T_2)\circ r(T_1)$.*

Proof

i) Requires a simple check. ii) From previous Theorem $r(T_2\circ T_1)$ and $r(T_2)\circ r(T_1)$ are covariant maps of $CP(\mathcal{S}_1)$. Let $\xi,\ \xi_2,\ \xi_1$ be the associated scale factors One has to show that $\xi = \xi_2\,\xi_1$. From equation (22) one finds indeed

$$\xi = \sum_{k=1}^N M(T_2\circ T_1)_{\mathbf{j}_1,\,\mathbf{i}_k} = \sum_{l=1}^M\sum_{k=1}^N M(T_2)_{\mathbf{j}_1,\,\mathbf{i}_l}\,M(T_1)_{\mathbf{i}_l,\,\mathbf{i}_k}$$
$$= (\sum_{l=1}^M M(T_2)_{\mathbf{j}_1,\,\mathbf{i}_l})\,(\sum_{k=1}^N M(T_1)_{\mathbf{i}_1,\,\mathbf{i}_k}) = \xi_2\,\xi_1, \tag{29}$$

where we used the independence of $M(T_2)_{\mathbf{i}_l,\,\mathbf{i}_k}$ on l. $\Box$

We conclude the section by a simple explicit computation, that shows the power of the notion of covariance. Let us consider the case $d = 2,\ N = 1,\ M = 2$, with initial state $\rho = 2^{-1}(\mathbf{I} + \sum_{\alpha=x,y,z}\lambda_\alpha\sigma_\alpha)$ (the σ's are the Pauli matrices). If $T \in CP_{2,1}$ is covariant and symmetric one must have $T(\mathbf{I}) \in \mathcal{A}_{2,2} = \mathrm{span}\{\mathbf{I}_2,\,C_2\}$ where

$$C_2 \equiv \sum_{\alpha=x,y,z}\sigma_\alpha\otimes\sigma_\alpha = 2\,P - \mathbf{I} \tag{30}$$

is a traceless combination of the identity and the transposition $P\,|\psi\rangle\otimes|\phi\rangle = |\phi\rangle\otimes|\psi\rangle$. Moreover the $T(\sigma_\alpha)$'s must be totally symmetric operators that transform according the adjoint ($j = 1$) representation of $SU(2)$. The totally symmetric sector of $\mathrm{End}(\mathbf{C}^4)$ is ten dimensional It is spanned by the elements of $\mathcal{A}_{2,2}$, ($j = 0$) five operators realizing a $j = 2$ multiplet of $SU(2)$, and by the $S_\alpha = 2\,\Delta_2(\sigma_\alpha)$, ($\alpha = x,y,z$) corresponding to $j = 1$. Therefore, from Theorem 2 [notice that trivially $\mathcal{H}_{sym}^1 = \mathcal{H}$] one has, $T(\sigma_\alpha) = \xi\,S_\alpha$. Putting all together $T(\rho) = 4^{-1}(\mathbf{I} + t\,C_2 + \xi\sum_\alpha\lambda_\alpha\,S_\alpha)$, one has

$$\mathrm{spec}\,T(\rho) = \{\frac{1}{4}(1\pm 2\,\xi + t),\ \frac{1}{4}(1 - 3\,t)\}$$

form which, by imposing the positivity *and* optimality, one immediately gets (by *covariance alone*) the Bužek-Hillery result $t = 1/3$ and $\xi_{max} = 2/3$ [3]. Notice that the optimal cloner has support in $\mathcal{H}_{sym}^2$.

IV. SUMMARY

In this note it has been rigorously shown that the optimal (with respect to a metric criterion) $N \mapsto M$ *pure state cloner* of a general d-dimensional quantum system can be described by a simple state-independent shrinking of the generalized Bloch vectors associated to the reduced density matrices. The structure of the proof can be summarized as follows. Over the space $CP_{M,N}$ of $N \mapsto M$ cloners a 'merit' functional is introduced in terms of the induced (non linear) maps of reduced (one-system) states. This functional –which has a clear geometrical meaning in the setting of the generalized Bloch representation (GBR) – is concave and invariant under the natural actions of the groups $S_M,\ SU(d)$. This allows us to restrict our attention to covariant (i.e. invariant respect to the group action) cloners: given a group orbit, by 'averaging' and using concavity, one can build a covariant cloner with no worse quality. This cloner results to be universal (cloning quality independent on the input state), and the components of the associated GBR map satisfy an automorphicity contraint. Allowing only for pure inputs and resorting to the intimate connection between representation theory of unitary and symmetric groups, one obtains the final result, that by linearity extends to the whole space of states over the totally symmetric subspace of $\mathcal{H}^{\otimes N}$.

ACKNOWLEDGMENTS

The author thanks D. Bruß, and C. Macchiavello for introducing him to the cloning problem, M. Rasetti for stimulating discussions and critical reading of the manuscript, Elsag-Bailey for financial support.

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